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Although chaotic systems with hidden attractors have been discovered recently, there are few investigations about relationships among them. This brief work introduces a novel chaotic system with only one stable equilibrium that is constructed by adding a tiny perturbation into a known chaotic flow having hidden attractors with a line equilibrium.

Keywords: Chaos; stable equilibrium; hidden attractor; circuit.

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1. Introduction

A possibility of finding a chaotic system with only one stable equilibrium has been presented by Wang and Chen [2012]. The striking discovery not only shows that there are still new mysterious features of chaos but also opens new attractive research directions [Wang & Chen, 2013; Kingni *et al.*, 2014; Lao *et al.*, 2014; Wei *et al.*, 2015a; Wei *et al.*, 2015b; Wei *et al.*, 2015c; Wei *et al.*, 2015d; Brezetskyi *et al.*, 2015; Dudkowski *et al.*, 2016]. For example, chaotic systems with an infinite number of equilibrium points [Jafari & Sprott, 2013; Gotthans & Petržela, 2015; Gotthans *et al.*, 2016], with one stable equilibrium [Molaie *et al.*, 2013; Wei & Zhang, 2014; Wei *et al.*, 2016] or chaotic systems without any equilibrium points [Wei, 2011; Wang *et al.*, 2012; Jafari *et al.*, 2013; Wei *et al.*, 2014] have been discovered recently. Interestingly, from the viewpoint of computation such systems can be considered as systems with "hidden attractors" [Leonov *et al.*, 2011a; Leonov *et al.*, 2011b; Leonov *et al.*, 2015b; Dudkowski *et al.*, 2016]. Systems with "hidden attractors" are important and have received considerable attention in recent years [Leonov *et al.*, 2014; Sharma *et al.*, 2015a, 2015b; Leonov *et al.*, 2015a; Wei *et al.*, 2015c; Shahzad *et al.*, 2015; Zhusubaliyev & Mosekilde, 2015].

Conventionally, the study of chaotic systems starts often with the study of equilibria. Systems with hidden attractors are proofs for this fact that the information on the number of equilibria and their types does not reflect, in itself, any difficulties in the study of the considered system (the study of equilibria, themselves, is not the problem). The challenging problems are finding attractors, explaining their formations, and the construction of systems with specific attractors, which are not necessarily related to the behavior of equilibria. It seems that investigating hidden attractors can help us to find the real answers of some open questions in chaos and nonlinear dynamics, since some old answers fail here.

Although the fact that different chaotic systems with hidden attractor have been reported, relationships among them should be studied further. One of the most interesting relationships is how we can achieve a new chaotic system with hidden attractors from another known one with hidden attractors. To the best of our knowledge, there is little information related to this topic in the literature. For example, a chaotic system having an infinite number of equilibrium points has been converted into another new chaotic system without equilibrium [Pham *et al.*, 2014] as illustrated in Fig. 1.

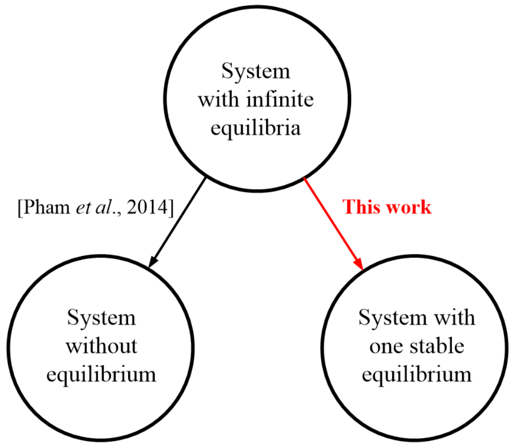


Fig. 1. Three main families of chaotic systems with hidden attractors (chaotic ones with only one stable equilibrium, with infinite equilibria, and without equilibrium) and their relationships.

In this work, we introduce a new chaotic system with only one stable equilibrium. Especially, such new system is constructed from a system with infinite equilibria. In addition, its dynamics and circuit implementation are also investigated.

2. New System with Only One Stable Equilibrium

We consider the simple chaotic flow LE_1 [Jafari & Sprott, 2013], which is described by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + yz, \\ \dot{z} = -x - axy - bxz, \end{cases} \tag{1}$$

in which x, y, z are state variables while a, b are positive parameters. The system (1) has a line of equilibria $E(0, 0, z)$ and can generate chaos [Jafari & Sprott, 2013].

By applying a tiny perturbation to the system (1), we can change its number of equilibrium points while preserving its chaotic dynamics [Wei, 2011; Wang & Chen, 2012]. Therefore, a new system is constructed by adding a simple parameter c into the system (1) as follows

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + yz + c, \\ \dot{z} = -x - axy - bxz, \end{cases} \tag{2}$$

where a, b are positive parameters $(a, b > 0)$ and c is the real parameter.

Obviously, the new system (2) becomes the system with an infinite number of equilibrium (1) for $c = 0$. However, there is the presence of countable equilibrium points in the system (2) for $c \neq 0$. We concentrate on this case.

The equilibrium of the system (2) is found by solving

$$y = 0, \tag{3}$$

$$-x + yz + c = 0, \tag{4}$$

$$-x - axy - bxz = 0. \tag{5}$$

From Eqs. (3)–(5), it is easy to see that the new system (2) has only one equilibrium $E(c, 0, -\frac{1}{b})$.

By linearizing the system (2) at the equilibrium E, we get the Jacobian matrix

$$\mathbf{J}_E = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -\dfrac{1}{b} & 0 \\ 0 & -ac & -bc \end{bmatrix}. \tag{6}$$

Therefore, the characteristic equation at the equilibrium point E is

$$(\lambda + bc)\left(\lambda^2 + \frac{1}{b}\lambda + 1\right) = 0. \tag{7}$$

We can rewrite Eq. (7) in another form:

$$\lambda^3 + f_z\lambda^2 + f_y\lambda + f_x = 0, \tag{8}$$

in which $f_z = bc + \frac{1}{b}$, $f_y = 1 + c$, and $f_x = bc$. According to the Routh–Hurwitz stability criterion, the equilibrium is stable for $a > 0$, $b > 0$, $c > 0$ since $f_z > 0$, $f_y > 0$, $f_x > 0$, and $f_z f_y > f_x$. In other words, we can obtain a system with only one stable equilibrium from the system with infinite equilibria by changing the value of the parameter c. In the next section, we report the chaoticity of the system (2) with only one stable equilibrium.

3. Chaotic Behavior of the System with Only One Stable Equilibrium

When $a = 15$, $b = 1$, and $c = 0.001$, the system (2) has only one stable equilibrium $E(0.001, 0, -1)$ with three eigenvalues $\lambda_1 = -0.001$, $\lambda_{2,3} = -0.5 \pm 0.866i$. Interestingly, the system is chaotic although having only one stable equilibrium. Its chaotic attractors are presented in Fig. 2. The well-known Wolf's method [Wolf *et al.*, 1985] has been applied to calculate the Lyapunov exponents. We have used the initial point $(x(0), y(0), z(0)) = (0, 0.5, 0.5)$ and the computational time $T = 10\,000$. Lyapunov

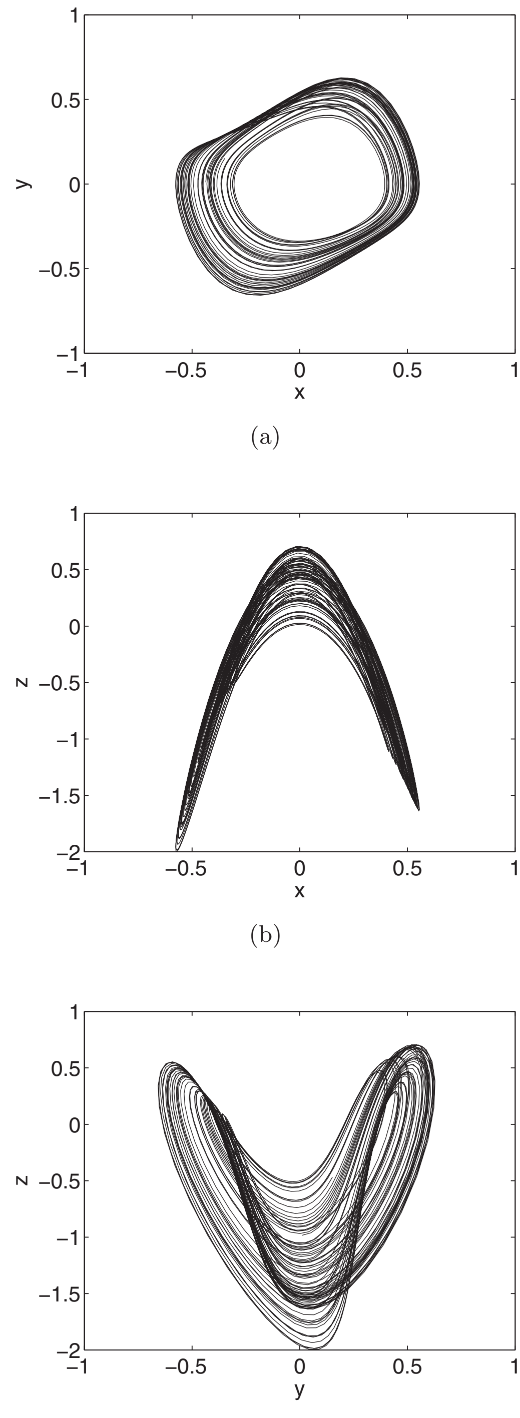


Fig. 2. Phase portraits of the system with only one stable equilibrium (2) in (a) x–y plane, (b) x–z plane and (c) y–z plane for $a = 15$, $b = 1$, $c = 0.001$, and the initial condition $(x(0), y(0), z(0)) = (0, 0.5, 0.5)$.

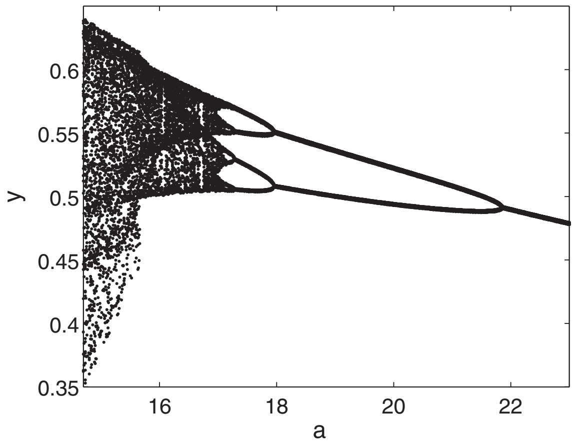


Fig. 3. Bifurcation diagram of the system with only one stable equilibrium when varying the bifurcation parameter a for $b = 1$ and $c = 0.001$.

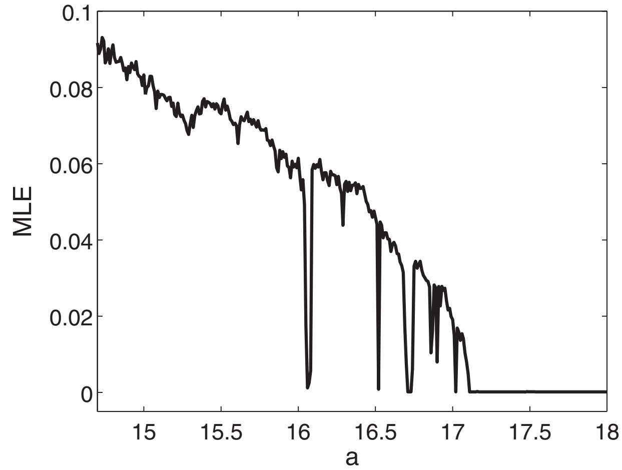


Fig. 5. Maximum Lyapunov exponents of the system with only one stable equilibrium versus the parameter a.

exponents and local Kaplan–Yorke dimension of the trajectory are $L_1 = 0.0833$, $L_2 = 0$, $L_3 = -0.5331$, and $D_{\mathrm{KY}} = 2.1563$, respectively. One may claim Lyapunov exponents are similar for almost all initial points on an attractor based on the ergodicity of the system. However, it is worth noting that the rigorous verification of the system's ergodicity is a challenging task. As a result, we should consider a grid of initial points on the attractor, corresponding to Lyapunov exponents as well as local Lyapunov dimensions [Kuznetsov, 2016; Kuznetsov *et al.*, 2016a; Leonov *et al.*, 2016]. Therefore, the Lyapunov dimension of the attractor is equal to the maximum of the corresponding local dimensions. Furthermore, an additional question is how to interpret the values of Lyapunov exponents and local Lyapunov dimensions, which should be investigated further.

In order to show the dynamical behavior of the system (2), we have decreased the value of the parameter a for $b = 1$ and $c = 0.001$. A route from period-doubling limit cycles to chaos is observed in Fig. 3. The state of the system is periodic for $a > 17.5$. However the system's states are complex for the range $a < 17$ (see Figs. 4 and 5) in which it can exhibit chaotic behavior. It is worth noting that the Shilnikov method cannot be applied to verify chaos in this case [Shilnikov, 1965; Shilnikov *et al.*, 1998; Dudkowski *et al.*, 2016] and chaotic attractor is hidden [Leonov *et al.*, 2011a; Leonov *et al.*, 2011b; Leonov *et al.*, 2015b; Dudkowski *et al.*, 2016].

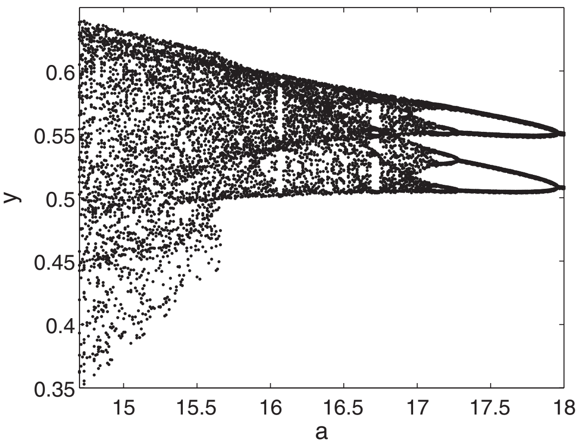


Fig. 4. Bifurcation diagram of the system with only one stable equilibrium when decreasing the value of parameter a from 18 to 14.7 for $b = 1$ and $c = 0.001$.

4. Discussion

The system (2) is a universal example to illustrate the presence of self-excited attractors and hidden attractors [Leonov *et al.*, 2011a; Leonov *et al.*, 2011b; Leonov *et al.*, 2015b] (see Table 1). As shown in Table 1, the system (2) displays self-excited chaotic attractor for the case A. The system (2) can exhibit different hidden attractors such as hidden chaotic attractors with an infinite number of equilibrium points (case B) and hidden chaotic attractors with only one stable equilibrium (case C).

Furthermore, we can design an electronic circuit to implement the proposed system (2). Figure 6 presents the schematic of our designed circuit. The circuit includes four operational amplifiers (U_1–U_4), three AD633 analog multipliers (U_5–U_7), 11 resistors and three capacitors. The voltages at the

Table 1. Three cases with the initial condition $(x(0), y(0), z(0)) = (0, 0.5, 0.5)$.

Case	Parameters	Equilibria	Eigenvalues	Lyapunov Exponents	Kaplan–Yorke Dimension
A	$a = 15$ $b = 1$ $c = -0.001$	$E(-0.001, 0, -1)$	$\lambda_1 = 0.001$ $\lambda_{2,3} = -0.5 \pm 0.866i$	$L_1 = 0.0728$ $L_2 = 0$ $L_3 = -0.5403$	$D_{\mathrm{KY}} = 2.1347$
B	$a = 15$ $b = 1$ $c = 0$	$E(0, 0, z)$	$\lambda_1 = 0$ $\lambda_{2,3} = \frac{1}{2}(z \pm \sqrt{z^2 - 4})$	$L_1 = 0.0717$ $L_2 = 0$ $L_3 = -0.5232$	$D_{\mathrm{KY}} = 2.1371$
C	$a = 15$ $b = 1$ $c = 0.001$	$E(0.001, 0, -1)$	$\lambda_1 = -0.001$ $\lambda_{2,3} = -0.5 \pm 0.866i$	$L_1 = 0.0833$ $L_2 = 0$ $L_3 = -0.5331$	$D_{\mathrm{KY}} = 2.1563$

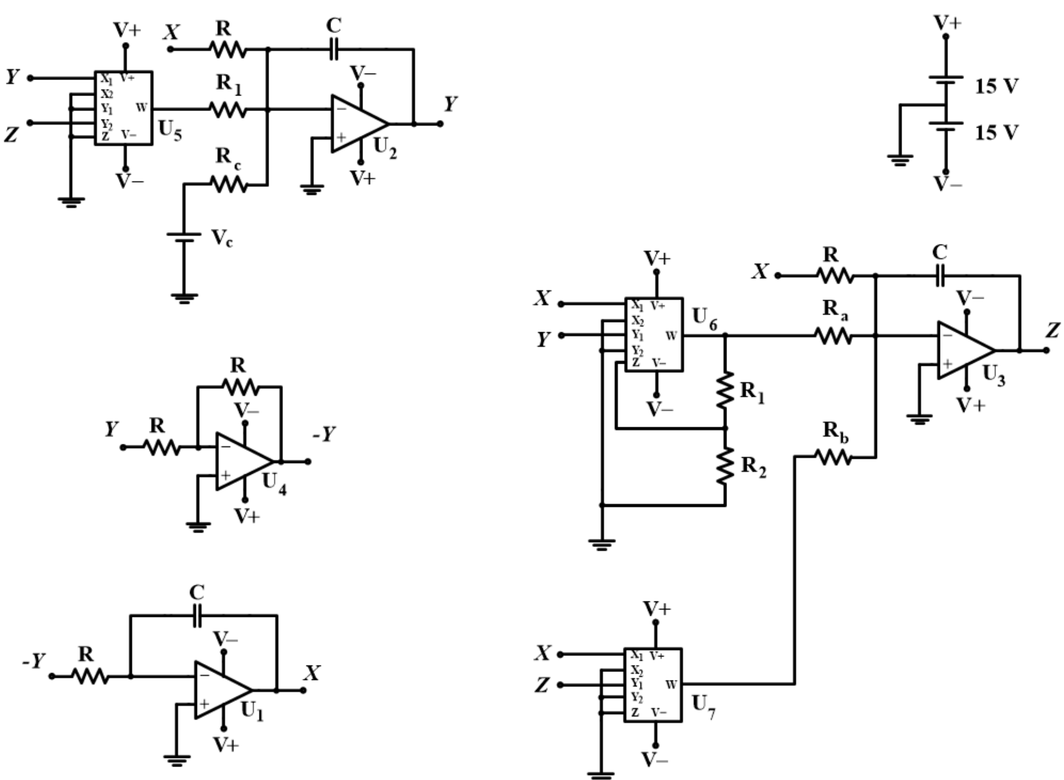
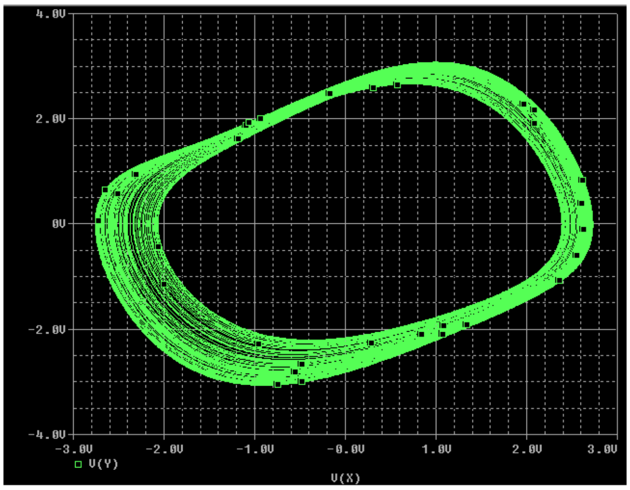
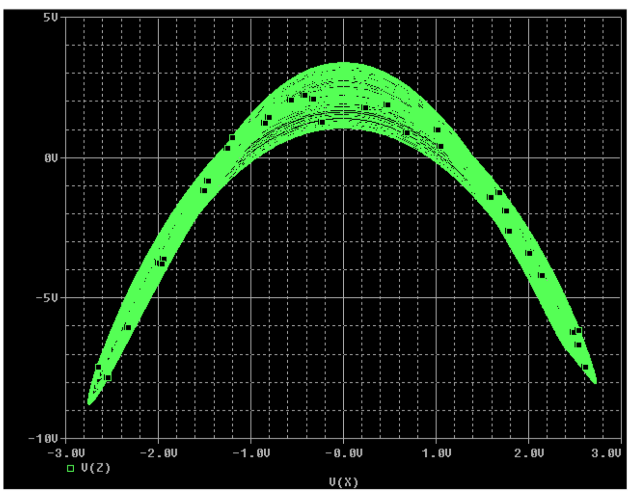


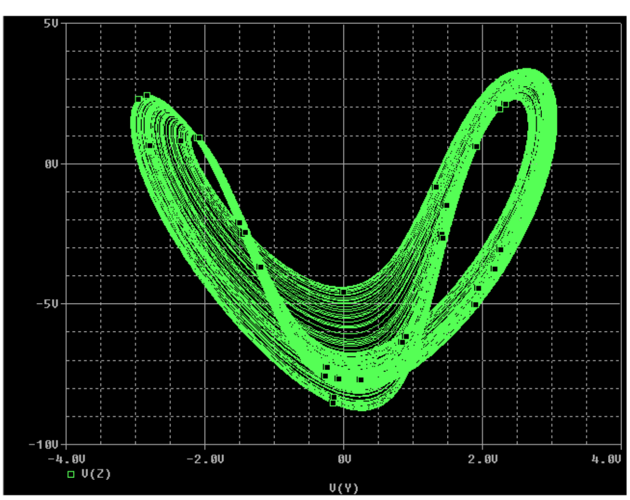
Fig. 6. Schematic of the circuit including common electronic components.



(a)



(b)



(c)

Fig. 7. Obtained PSpice phase portraits from the circuit in (a) X–Y plane, (b) X–Z plane and (c) Y–Z plane.

outputs of operational amplifiers are denoted as X, Y, and Z.

The circuital equations are described by

$$\begin{cases} \dot{X} = \dfrac{1}{RC}(Y), \\ \dot{Y} = \dfrac{1}{RC}\left(-X + \dfrac{R}{R_1 10\,\mathrm{V}}YZ - \dfrac{R}{R_c}V_c\right), \\ \dot{Z} = \dfrac{1}{RC}\left(-X - \dfrac{R}{R_a}\dfrac{R_1 + R_2}{R_1 10\,\mathrm{V}}XY - \dfrac{R}{R_b 10\,\mathrm{V}}XZ\right). \end{cases} \tag{9}$$

By normalizing the circuital equation (9) with $\tau = \frac{t}{RC}$, it is equivalent to the system (2). The circuit is implemented in the Cadence OrCAD with the values of components as follows: $R = 20\,\mathrm{k\Omega}$, $R_1 = 10\,\mathrm{k\Omega}$, $R_2 = 10\,\mathrm{k\Omega}$, $R_a = 1.333\,\mathrm{k\Omega}$, $R_b = 10\,\mathrm{k\Omega}$, $R_c = 20\,\mathrm{k\Omega}$, $V_c = -5\,\mathrm{mV_{DC}}$ and $C = 4.7\,\mathrm{nF}$. PSpice results are reported in Fig. 7. The similarity between the theoretical attractors (Fig. 2) and the circuital ones (Fig. 7) shows the feasibility of the theoretical system (2). Engineering applications of such system will be discovered in our future researches due to its chaos and feasibility. It is noted that there are various software simulators of circuits such as Cadence OrCAD, SPICE, MATLAB Simulink etc. However, results of such software simulators depend on numerical procedures implemented in the software and may be qualitatively different from the physical behavior of the considered circuit [Bianchi *et al.*, 2015; Bianchi *et al.*, 2016; Kuznetsov *et al.*, 2016b]. Therefore, considerable attention must be paid when using these software simulators [Bianchi *et al.*, 2015].

5. Conclusions

This work contributes to the knowledge of chaotic systems with hidden attractors. The proposed system also illustrates the conversion of a hidden attractor, which is rarely reported. We believe that such system can be used to study a route from self-excited attractors to hidden attractors.

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