FIR FILTER DESIGN USING A NEW WINDOW FUNCTION

Mahrokh G. Shayesteh [1] and Mahdi Mottaghi-Kashtiban [1,2]

[1]Department of Electrical Engineering, Urmia University, Urmia, Iran
[2]Sonar Seraj System Cor., Urmia, Iran

Emails: m.shayesteh@urmia.ac.ir ; st-mmottaghi@urmia.ac.ir

ABSTRACT

We present a new simple form window function with application to FIR filter design, which has superior performance compared to the several commonly used windows. The new window is almost equi-ripple. For sufficiently high orders, the proposed window has the main lobe width of roughly equal to that of the commonly used Hamming window, while featuring 5.6~12 dB smaller side lobe peak. For lower window lengths, the performance of Hamming window is degraded, while the new window maintains its maximum side lobe peak about -48~-49.5 dB compared to -37~ -39 dB of Hamming window, and at the same time offers smaller main lobe width. Our results also show substantial performance improvement of the proposed window compared to the Gaussian, Kaiser, and Lanczos windows. A performance comparison of the proposed and Dolph-Chebyshev windows is also presented. The designed FIR filters confirm the efficiency of the new window.

Keywords: FIR filter, Hamming, Kaiser, Dolph-Chebyshev, Parks-McClellan algorithm.

I. INTRODUCTION

Since the idealized discrete-time filters are usually defined by piece-wise constant or piece-wise functional frequency responses with discontinuities at the boundaries, they have non-causal infinite impulse responses (IIR). The most straightforward approach to obtain a causal finite impulse response (FIR) is to truncate the ideal response. If $h_d[n]$ is the impulse response of the desired (ideal) IIR system, the simplest way to obtain a casual FIR filter is to define a new system with impulse response $h[n]$ of length $M+1$ as:

$$h[n] = h_d[n]\, w[n] \quad ; \quad w[n] = \begin{cases} f(n) & 0 \le n \le M \\ 0 & otherwise \end{cases} \quad (1)$$

where $f(n)$ is a function of n. In other words, we can represent $h[n]$ as the product of the desired response $h_d[n]$ and a finite-duration "window", $w[n]$. Therefore, the Fourier transform of $h[n], H(e^{jw})$, is the periodic convolution of the desired frequency response, $H_d(e^{j\omega})$, with Fourier transform of the window, $W(e^{jw})$. As a result,

$H(e^{jw})$ will be a smeared version of $H_d(e^{j\omega})$. Two desirable specifications for a window function are smaller main lobe width and good side lobe rejection (smaller ripple ratio). However these two requirements are contradictory, since for a given length, a window with a narrow main lobe has a poor side lobe rejection and vice versa. The simplest window is the rectangular window with $f(n)=1$ in eq. (1). This window has a very sharp transition band (small main lobe width); however, its relative peak of the side lobe is approximately -13 *dB* [1] (the highest peak among the commonly used windows), which causes large ripples in $H(e^{jw})$. This can be moderated through the use of a less abrupt truncation of $h_d[n]$, i.e. by tapering the window smoothly to zero at each end. Various shapes of window functions have been introduced which have different characteristics; some of them are: Bartlett, Hanning, Hamming, Blackman, Lanczos, Kaiser, Gaussian, and Dolph-Chebyshev windows [1-3]. For the equal length of *M*+1, the first three windows have approximately the same main lobe width; among them, the Hamming window offers the smallest peak of side lobe. The Blackman window has smaller side lobe peak compared to the Hamming window, but its main lobe width is wider; therefore they are employed in different applications depending on the required specifications, and can not be fairly compared. Lanczos window [2], also known as "Sinc" window, is the main lobe of a Sinc function powered by a positive integer, and therefore can show different characteristics. The Kaiser and Gaussian windows are tunable functions, and offer the property that their main lobe widths and side lobe peaks, in a trade-off, can be customized. The Kaiser window has the disadvantage of higher computational complexity due to the use of Bessel functions in the calculation of the window coefficients. Dolph-Chebyshev window [3] can be expressed as a cosine series which has high cost of computation, but its important property is that all side lobes are equal and the main lobe width is the minimum that can be achieved for a given ripple ratio. There has been great interest into the design of new windows to meet the desired specification for different applications [4-6].

In this paper, we present a new window function which has at least 5.6~12 *dB* less side lobe peak compared to the Hamming window, while offering smaller or equal main

lobe width. We also show that the new window is better than the other windows such as Hanning, Bartlett, Gaussian, Kaiser, and Lanczos. Further, we compare the performance of the proposed window with the Dolph-Chebyshev window. We design FIR filter with the new window to evaluate its efficiency.

II. PROPOSED WINDOW

In some applications (FIR filters, FFT, signal processing and measurements), it is necessary to reduce the level of side lobes below -45 *dB*. For a window length of $M+1$, the Hamming window gives good main lobe width (approximately $8\pi/M$) and side lobe level (about -42 *dB*), which is not low enough. On the other hand, the Blackman window offers very low side lobe peak (-58 *dB*), but its main lobe is wider ($12\pi/M$). Tunable windows such as Kaiser and Dolph-Chebyshev are used to give the desired main lobe width-side lobe peak trade off. But, the problem of tunable windows is the high cost of computations.

The goal of this work is to find a window with simple closed form formula, having equal main lobe width and smaller side lobe peak compared to the Hamming window. An optimization algorithm, the Extended Kalman Filter (EKF), is used in obtaining the new window.

We define window coefficients as the state of the EKF and let the EKF optimize them for the desired performance. At first, we choose the coefficients of Hamming window as the initial state and then we define an error function using the FFT of the window as

$$E = \frac{1}{S^2} \sum_{m=2}^{S} \left[P(1) - P(m) \right]^2 \tag{2}$$

where S is the number of side lobes and $P(m)$ is the peak of the mth side lobe. The EKF tends to minimize the error (E), which means that the resulted window will be equi-ripple if $E=0$ (note that the equi-ripple window results in optimum window [1-2]). We can model this procedure as a system with the following specifications:

$$x_{k+1} = x_k + w_k \quad ; \quad d_k = E_k + v_k \tag{3}$$

where x_k is the state of the EKF at the kth iteration (the obtained coefficients), E_k is the error, w_k is the system noise with covariance of Q, and v_k is the measurement noise of covariance R.

Starting with an initial estimation of the optimized state (x_0), the EKF can find the optimum state as follows

$$H_k = \frac{\partial E}{\partial x}\Big|_{x=\hat{x}_{k-1}} \quad \Rightarrow \quad G_k = P_k H_k^T \left(R + H_k P_k H_k^T \right)^{-1} \tag{4}$$

$$\hat{x}_k = \hat{x}_{k-1} - G_k E_{k-1} \quad \Rightarrow \quad P_{k+1} = P_k - G_k H_k P_k + Q \tag{5}$$

where $\frac{\partial E}{\partial x}$ is calculated by numerical methods, G_k is known as the Kalman gain, and P_k is the covariance matrix of the state estimation error. Initially, we assume that the optimal state is the coefficients of the Hamming window which has the error with covariance P_k. At first, we set $P_0 = aI$ where a is a constant selected by trial and error and $I_{(M+1)\times(M+1)}$ is identity matrix ($M+1$ is the number of filter coefficients). The iterations of EKF begins with $k=1$ where $P_1 = P_0 + Q$. In each step, P_k is updated as (5).

After several iterations, the optimum coefficients are obtained. We found that with varying M, the first and last coefficients ($w[0]$ and $w[M]$) have different behavior than other coefficients; that is, the shape of the window for $n=1…M$-1 is not considerably changed, but $w[0]=w[M]$ sometimes show abrupt changes. Hence, we tried to present the dependence of $w[0]=w[M]$ on M with formula different from the one used for $0<n<M$. We looked for closed form formulas that matches the obtained coefficients. We have examined several formulas and finally we found that the Sinc function as below is a good one that fits the optimum coefficients for $0<n<M$. The coefficients $w[0]=w[M]$ noting the above discussions are calculated separately. In this way, the simplest formula we found was as:

$$w[n] = \begin{cases} 0.02 + 0.001M + \dfrac{1}{2M+50} & n = 0, M \\ Sinc^{2.5}\left(\dfrac{n-M/2}{0.654M}\right) & 0 < n < M \\ 0 & otherwise \end{cases} \tag{6}$$

where $Sinc[n] = \dfrac{\sin(\pi n)}{\pi n}$. Note that the new window has the property that:

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & otherwise, \end{cases} \tag{7}$$

i.e. it is symmetric about the point $M/2$ and consequently has a generalized linear phase.

Fig. 1 shows the shape of the new window for different values of M. For $M<38$, $w[0]=w[M]$ are smaller than $w[1]=w[M-1]$, respectively and the window is like a bell-shaped function. For higher values of M, $w[0]$ and $w[M]$ are larger than $w[1]$ and $w[M-1]$, respectively. Our results show that the main lobe of new window is about $8\pi/M$ which is equal to that of Hamming window.
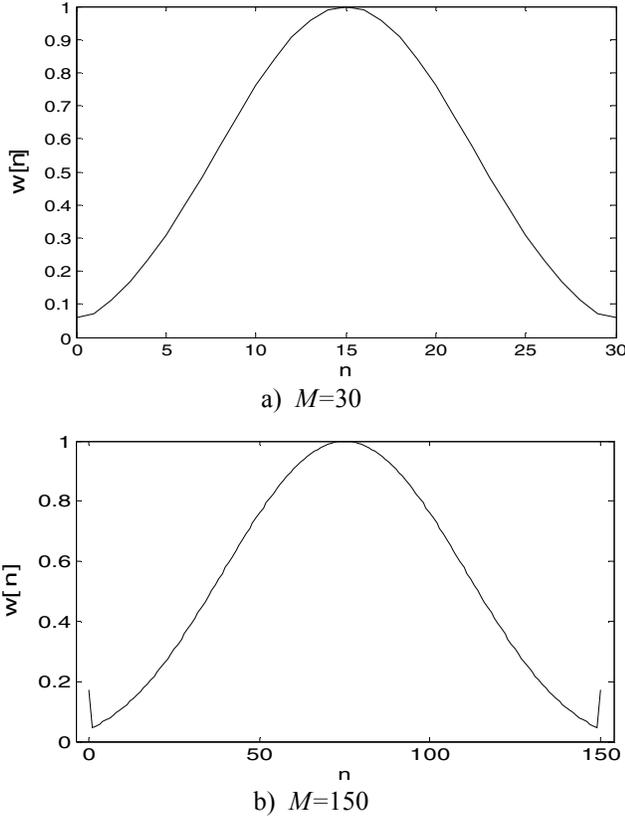
a) M=30

b) M=150

Fig. 1. Shape of the proposed window for different lengths.

a) M=50

b) M=14

Fig. 2. Fourier transforms of the proposed and Hamming windows for different lengths.

III. PERFORMANCE ANALYSIS

In this section, we compare the performance of the proposed window with several commonly used windows.

A- Hamming window

The Hamming window is widely used in different applications such as speech processing; it has the shape of

$$w_H[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n / M) & 0 \le n \le M \\ 0 & else. \end{cases} \quad (8)$$

Fig. 2.a shows Fourier transforms of the Hamming and proposed windows for M=50. We observe from this figure that the two windows have the same main lobe width $(2 \times 0.082\pi)$, while the proposed window offers 5.7 dB smaller peak of side lobe (-48 dB compared to -42.3 dB of the Hamming window). For lower window lengths, the performance of the Hamming window is degraded. For example, Fig. 2.b shows the Fourier transforms for M=14. It is observed that the Hamming window has the side-lobe peak of -39 dB and main lobe width of $2 \times 0.322\pi$, while the proposed window has side lobe peak of -49.7 dB and main lobe width of $2 \times 0.298\pi$. The reduction in side lobe peak is 12.3 dB for M=10 (-49 dB compared to -36.7 dB).
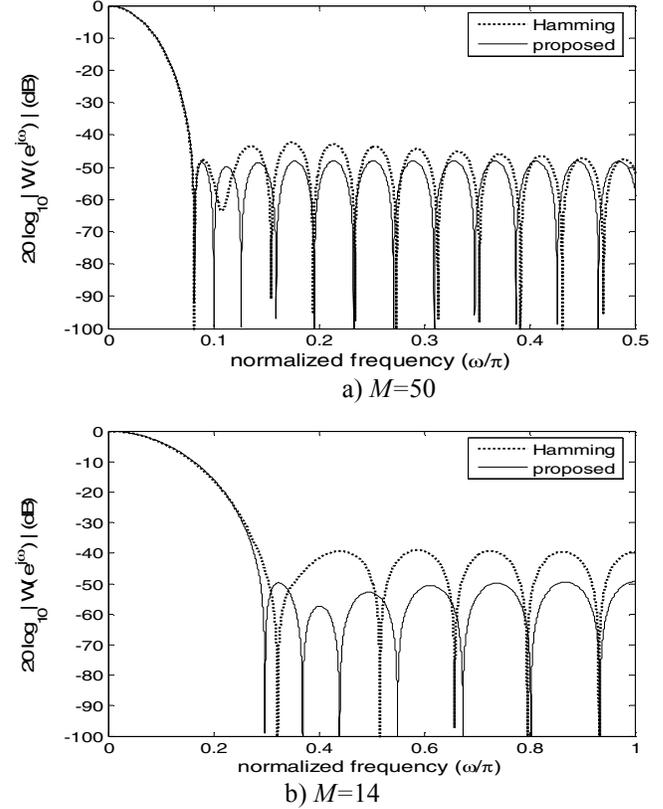
Our analysis shows that for $10 \le M \le 1000$, the side lobe peak of the proposed window compared to that of the Hamming window is 5.6~12.5 dB smaller. So, the proposed window has less than or equal main lobe to that of the Hamming window while offering less side lobe peak. Since Hamming window has the same main lobe width as Bartlett and Hanning windows while giving less side-lobe peak (about -41 dB compared to -26 dB and -31 dB, respectively), it is obvious that, in the case of side lobe peak, the proposed window is also better than Bartlett and Hanning windows.

B- Kaiser window

The Kaiser window has the following shape:

$$w_K[n] = \begin{cases} \dfrac{I_0\left[\beta\left(1 - [(n - M/2)/(M/2)]^2\right)^{0.5}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & else \end{cases} \quad (9)$$

where β is the tuning parameter of the window to obtain the desired "main lobe width – side lobe peak" tradeoff, and $I_0(.)$ is the zero order modified Bessel function of the first kind.

Fig. 3 demonstrates the performance of the proposed and Kaiser windows for two different values of β and the same length of M+1=51. From Fig. 3.a, it is observed that for

β=6.55, the two windows have the same side lobe peak (-48 *dB*) while the proposed window gives less main lobe width. If we want the Kaiser window to have the same main lobe of the new window, then its length should be *M*+1=57. Therefore, the proposed window offers the desired specifications with lower length.

However, we can decrease β to 5.61 if we consider the same main lobe width and *M*; but in this case, the proposed window has less side lobe peak (-48 *dB* compared to -41.4 *dB*), as we can see from Fig. 3.b.
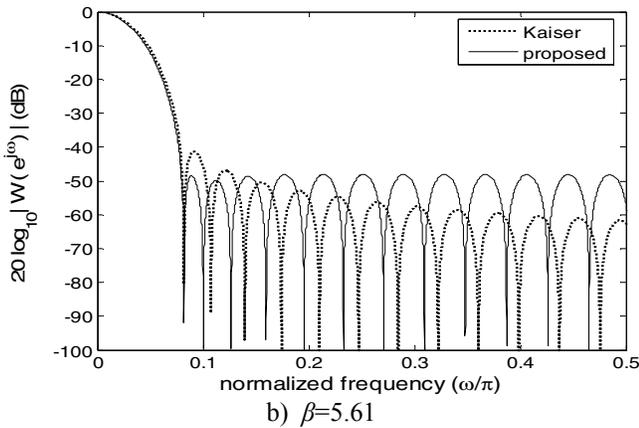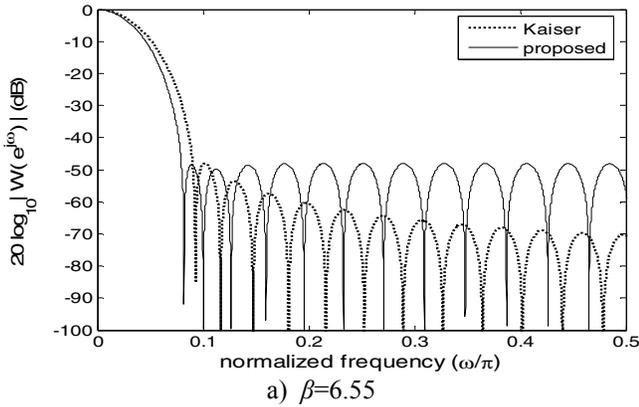

a) β=6.55

b) β=5.61

Fig. 3. Fourier transforms of the new and Kaiser windows, *M*=50.

C- Gaussian window

The Gaussian window is of the form

$$w_G[n] = \begin{cases} e^{-\frac{1}{2}\left(\frac{n-M/2}{\sigma M/2}\right)^2} & 0 \le n \le M \\ 0 & otherwise \end{cases} \quad (10)$$

where $\sigma<0.5$ is the tuning parameter of the window to have the desired "main lobe width – side lobe peak" tradeoff. By setting σ=0.375 for this window, Fig. 4 depicts that the side lobe peak of the two windows is -48 *dB* (for *M*=50), while the main lobe width of the proposed window is much less than that of the Gaussian window. Our results show that, for the equal main lobe width, the new window has 12 *dB* less side lobe peak.
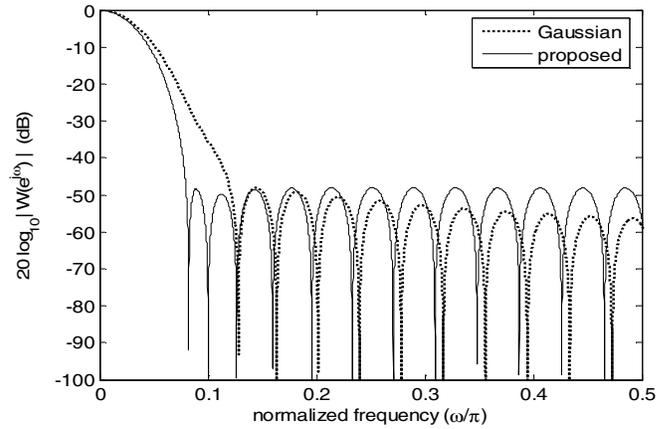

Fig. 4. Fourier transforms of the proposed and Gaussian (σ=0.375) windows for *M*=50.

D- Dolph-Chebyshev window

This window can be expressed as a cosine series [3]:

$$w[n] = \frac{1}{M+1}\left[T_M(x_0) + \sum_{i=1}^{M/2} T_M\left(x_0 \cos\frac{i\pi}{M+1} \right) \cos\left(\frac{2i\pi}{M+1} \right) \right] \quad (11)$$

where $T_M(x)$ is the Chebyshev polynomial of degree *M* and x_0 is a function of side lobe peak and *M*. Dolph-Chebyshev window has high cost of computation, but its important property is that all side lobes are equal and the main lobe width is minimum that can be achieved for a given ripple ratio. From Fig. 5, we observe that, the new window has a little less performance than Dolph-Chebyshev window (0.45 *dB* higher side lobe peak). But, note that the new window coefficients can be computed easier, and as we will see later, the FIR filter obtained from the new window has even better minimum stopband attenuation than the filter resulted from Dolph-Chebyshev window.
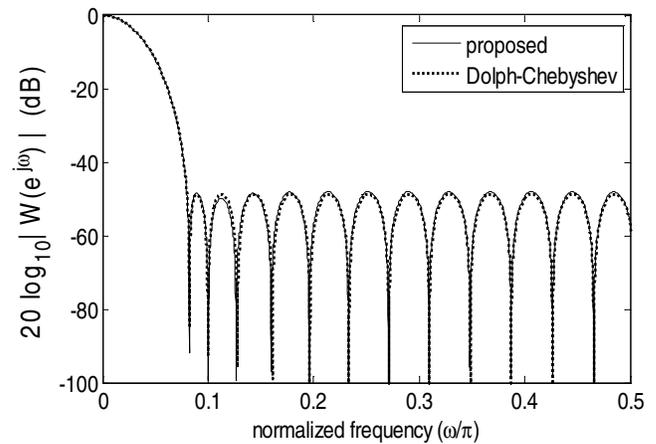

Fig. 5. Fourier transforms of the new and Dolph-Chebyshev windows, *M*=50.

E- Lanczos window

The Lanczos window has the form of:

$$w_L[n] = \begin{cases} Sinc^L\left(\dfrac{2n}{M}-1\right) & 0 \le n \le M \\ 0 & otherwise \end{cases} \qquad (12)$$

where L is a positive integer number. For M=50, Figs 5.a and 5.b compare the proposed and Lanczos windows for L=1 and 2, respectively. Fig. 5.a shows that the side lobe peak of the new window is 22 dB less than that of the Lanczos window, while its main lobe is just a little wider $(2 \times 0.017\pi)$; therefore they can not be fairly compared. From Fig. 5.b, we find that for L=2, the new window has both less main lobe width $(2 \times 0.082\pi$ compared to $2 \times 0.094\pi)$ and side lobe peak (-48 dB compared to -39.6 dB).
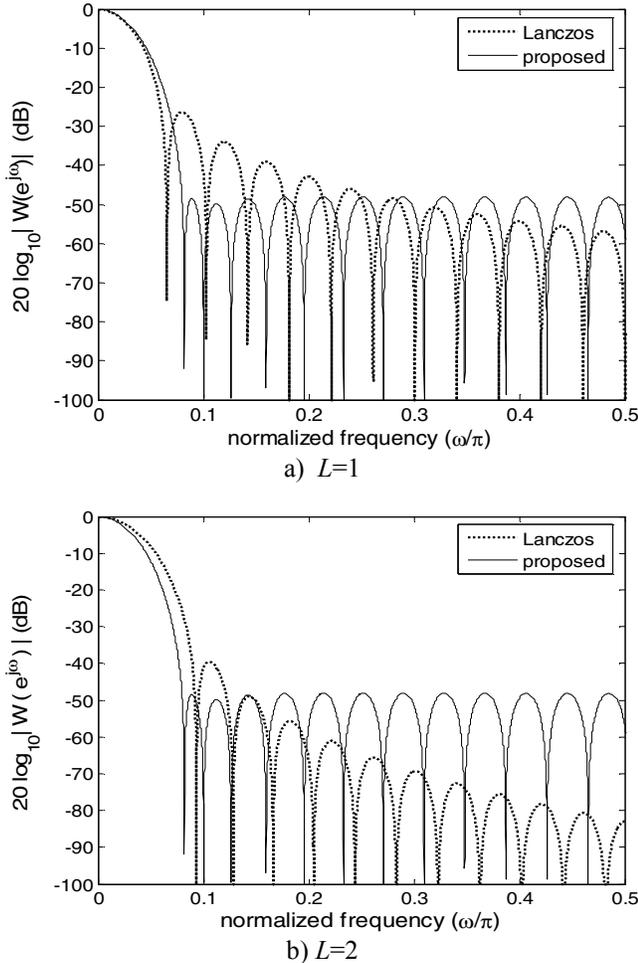

a) L=1

b) L=2

Fig. 5. Fourier transforms of the new and Lanczos windows, M=50.

IV. APPLICATION EXAMPLE

In order to show the efficiency of the proposed window and compare the results with the other windows, an example of designing an FIR low pass filter by windowing of an ideal IIR low pass filter is considered. Having a cut-off frequency of ω_C, the impulse response of an ideal low pass filter is:

$$h_{LP,Ideal}[n] = \frac{\sin(\omega_C n)}{\pi n} \qquad (13)$$

By windowing this IIR filter with the windows discussed in the paper, different FIR filters can be obtained.

For ω_C=0.2π, Figs. 7.a-7d show the frequency response of the FIR filters designed by applying different windows of length M+1=51. We observe that the filter achieved by the new window has 9 dB less side lobe peak than what can be obtained by using the Hamming window. These values are about 2.8 dB for Kaiser, and 8 dB for Gaussian windows. Also the FIR filter designed with the proposed window has 1.1 dB smaller side lobe level than that of achieved by Dolph-Chebyshev window.

Further, Fig. 7.e demonstrates that the filter obtained by Parks-McClellan algorithm [1-2] with the same specifications as the filter obtained from the new window results in M=48 which is a little less than M=50 of the proposed window. Note that the Parks-McClellan algorithm needs computer program while our window has a simple closed formula.

V. CONCLUSIONS

A novel efficient window having a closed simple formula was presented which is symmetric and almost equi-ripple. The proposed window was obtained by optimizing the coefficients of Hamming window using the extended Kalman filter. Performance comparison of the proposed window with the Hamming, Gaussian, Kaiser, and Lanczos windows shows that, for the same window length, if the proposed window has the same main lobe as the other windows, then it offers less side lobe peak and for the case of same side lobe peak, the new window gives less main lobe width. The average reduction in the side lobe peak of the proposed window compared to that of the Hamming, Kaiser, Gaussian, and Lanczos windows is 5.7 dB, 6.6 dB, 12 dB, and 8 dB, respectively. It is obvious that the proposed window also outperforms the Hanning and Bartlett windows. The FIR filter designed using the new window achieves less ripple ratio compared to the filters obtained using the above mentioned windows and Dolph-Chebyshev window which confirms the better performance of the proposed window. Further, for the same specification of FIR filter, the length of the proposed window is a little more than the length of the filter obtained by the Parks-McClellan algorithm, while the new window has simple form.

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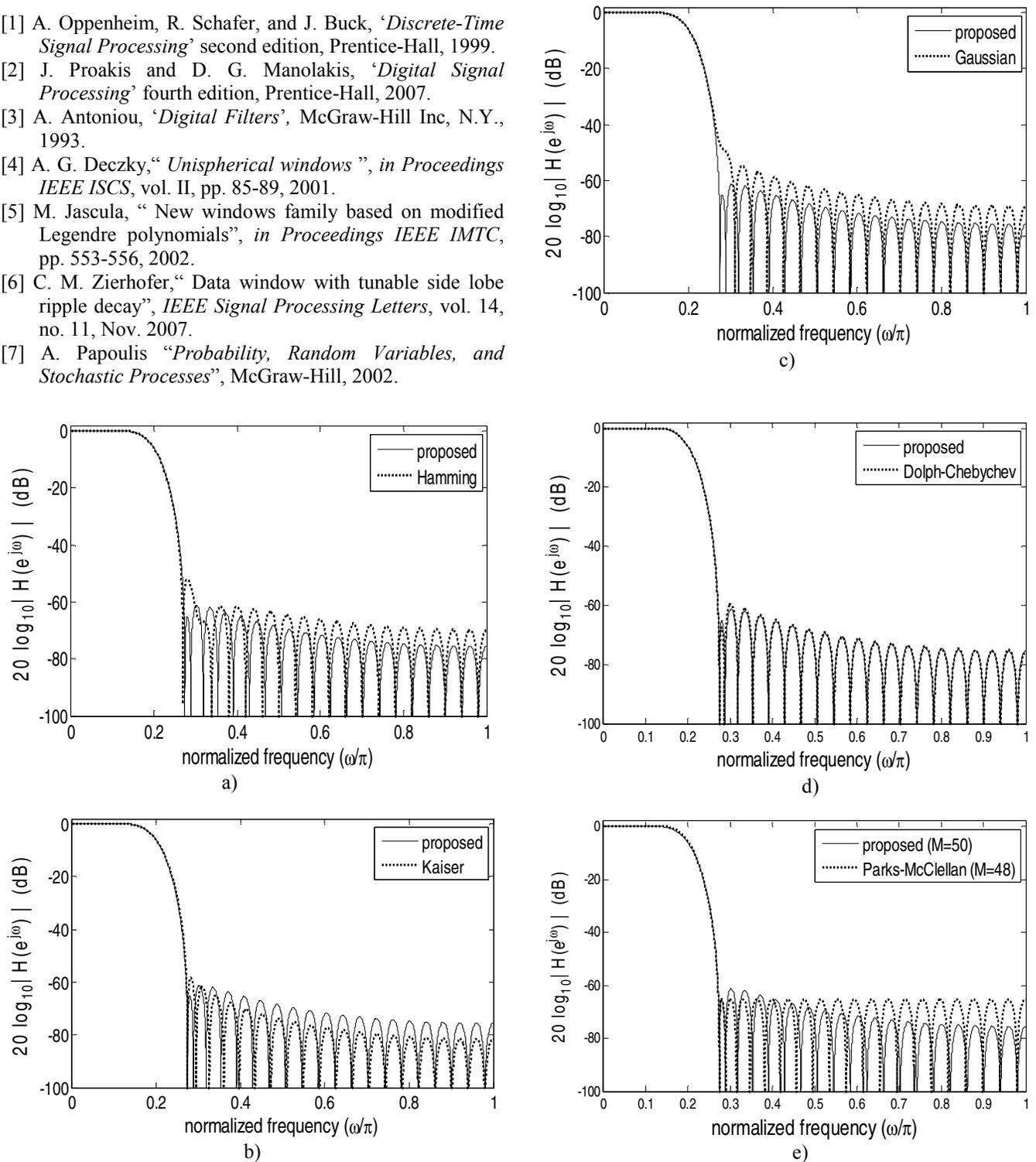[7] A. Papoulis "*Probability, Random Variables, and Stochastic Processes*", McGraw-Hill, 2002.

Fig. 7. Performance of FIR low pass filter, *M*=50, obtained by windowing of an IIR filter with different windows, and also using Parks-McClellan algorithm, $\omega_c = 0.2\pi$.